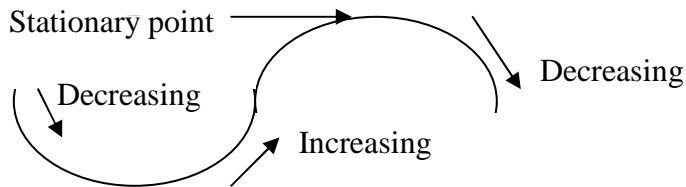


Increasing and Decreasing Functions

A positive gradient means the function is increasing $f'(x) > 0$

A negative gradient means the function is decreasing $f'(x) < 0$

When the gradient is zero then it is a stationary point $f'(x) = 0$



Example 1. For what values of x is the function $f(x) = 3x^2 + 4x - 3$ increasing?

$$f(x) = 3x^2 + 4x - 3$$

$$f'(x) = 6x + 4 \quad \text{for increasing function } f'(x) > 0$$

$$\therefore 6x + 4 > 0$$

$$6x > -4$$

$$x > -\frac{4}{6} \quad \text{or} \quad x > -\frac{2}{3}$$

so the function is increasing when $x > -\frac{2}{3}$

Example 2. For what values of x is the function $f(x) = 2x^3 - 3x^2 + 6$ decreasing?

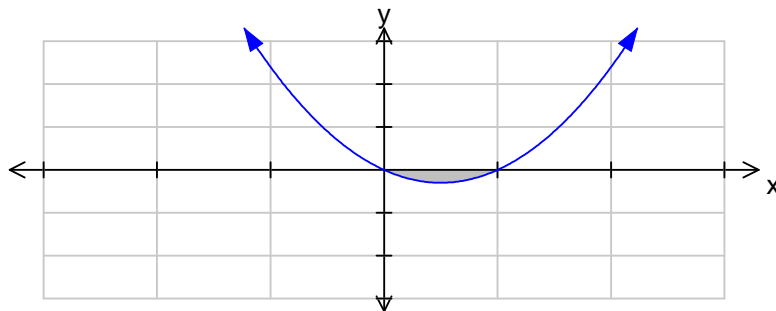
$$f(x) = 2x^3 - 3x^2 + 6$$

$$f'(x) = 6x^2 - 6x \quad \text{for decreasing function } f'(x) < 0$$

$$\therefore 6x^2 - 6x < 0$$

$$6x(x - 1) < 0$$

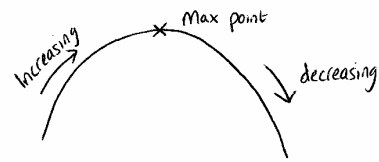
critical values are $x = 0$ and $x = 1$



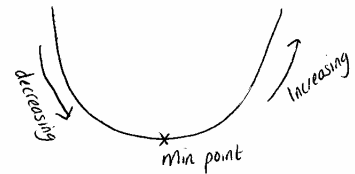
So the function is decreasing when $0 < x < 1$

Finding Coordinates of a Stationary Point

When $f(x)$ stops increasing and begins to decrease it is called a maximum point.



When $f(x)$ stops decreasing and begins to increase it is called a minimum point.



The maximum and minimum points are called **turning points**. At these points the gradient is zero.

That means that $f'(x) = 0$

Example 1.

Find the coordinate of the turning point of the function $f(x) = 5x^2 + 3x + 2$. State whether it is a maximum or minimum turning point.

$$f(x) = 5x^2 + 3x + 2$$

$$f'(x) = 10x + 3$$

$$f'(x) = 0 \quad \text{for a stationary point}$$

$$\therefore 10x + 3 = 0$$

$$10x = -3$$

$$x = -\frac{3}{10}$$

$$\text{when } x = -0.3 \quad y = 5x^2 + 3x + 2$$

$$y = 5 \times (-0.3)^2 + (3 \times -0.3) + 2$$

$$y = 0.45 - 0.9 + 2$$

$$y = 1.55$$

$$\therefore \text{the turning point is } (-0.3, 1.55)$$

$$\text{when } x = -1 \qquad x = -0.3 \qquad x = 0$$

$$f'(x) = -7 \qquad f'(x) = 0 \qquad f'(x) = 3$$

$$\backslash \qquad \qquad - \qquad \qquad /$$

by looking at the lines we can see that the turning point is a minimum

Another way to find out if it's a maximum or a minimum point is to calculate the 2nd derivative, as this measures the change in the gradient.

If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$ its a minimum point

If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$ its a maximum point

If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$ it could be a minimum, maximum or point of inflection

If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3} \neq 0$ then its a point of inflection

Example 2. Find the greatest value of $f(x) = 3 + 2x - x^2$

This means find the y coordinate of the maximum turning point

$$f(x) = 3 + 2x - x^2$$

$$f'(x) = 2 - 2x$$

$$f'(x) = 0$$

$$0 = 2 - 2x$$

$$2x = 2$$

$$x = 1$$

$$\text{when } x = 1 \quad y = 3 + 2x - x^2$$

$$y = 3 + (2 \times 1) - (1^2)$$

$$y = 3 + 2 - 1$$

$$y = 4$$

\therefore Greatest value of $f(x) = 4$

Example 3. Find the coordinates of the points where the gradient is zero, state where the points are max, min or a point of inflexion, for the function

$$f(x) = x^3 - x^2 - x + 1$$

$$y = x^3 - x^2 - x + 1$$

$$\frac{dy}{dx} = 3x^2 - 2x - 1$$

$$\frac{dy}{dx} = 0 \text{ for turning points}$$

$$\therefore 3x^2 - 2x - 1 = 0$$

$$(3x + 1)(x - 1) = 0$$

$$x = 1 \text{ or } x = -\frac{1}{3}$$

$$\begin{aligned} \text{when } x = 1 \quad y &= x^3 - x^2 - x + 1 \\ y &= 1^3 - 1^2 - 1 + 1 \\ y &= 0 \end{aligned} \qquad (1,0)$$

$$\text{when } x = -\frac{1}{3} \quad y = x^3 - x^2 - x + 1$$

$$y = \left(-\frac{1}{3}\right)^3 - \left(-\frac{1}{3}\right)^2 - \left(-\frac{1}{3}\right) + 1$$

$$y = -\frac{1}{27} - \frac{1}{9} + \frac{1}{3} + 1$$

$$y = 1\frac{5}{27} \quad \left(-\frac{1}{3}, 1\frac{5}{27}\right)$$

$$\text{considering } \frac{d^2y}{dx^2} = 6x - 2$$

$$\text{at } x = 1 \quad \frac{d^2y}{dx^2} = 4 \quad \therefore \frac{d^2y}{dx^2} > 0 \quad \therefore (1,0) \text{ is a minimum point}$$

$$\text{at } x = -\frac{1}{3} \quad \frac{d^2y}{dx^2} = -4 \quad \therefore \frac{d^2y}{dx^2} < 0 \quad \therefore \left(-\frac{1}{3}, 1\frac{5}{27}\right) \text{ is a maximum point}$$

Using Turning Points to Solve Problems

You approach a word problem in the same way as other problems. You can use differentiation to find the maximum or minimum area or volume in the same way as we calculated turning points.

Make sure you rearrange and substitute equations so that you end with an equation with only one variable.

Example 1.

A large cuboid tank is made from 54m^2 of sheet metal. The tank has a horizontal base but no lid. The height is x m and the vertical opposite faces are squares. Find the maximum volume.

let the length be y , the width be x and the height be x .

$$\text{volume} = x^2 y$$

$$\text{Surface area} = x^2 + x^2 + 3xy \quad \text{also SA} = 54\text{m}^2$$

$$54 = 2x^2 + 3xy$$

$$\frac{54 - 2x^2}{3x} = y$$

substitute y into the volume formula

$$v = x^2 y$$

$$v = x^2 \left(\frac{54 - 2x^2}{3x} \right)$$

$$v = \frac{54x^2 - 2x^4}{3x}$$

$$v = 18x - \frac{2x^3}{3}$$

To find the maximum volume we differentiate

$$v = 18x - \frac{2x^3}{3}$$

$$\frac{dv}{dx} = 18 - 2x^2$$

$$\frac{dv}{dx} = 0 \quad \text{for a turning point}$$

$$0 = 18 - 2x^2$$

$$2x^2 = 18$$

$$x^2 = 9$$

$$x = \pm 3$$

as x is a length x must be 3

$$\text{when } x = 3 \quad v = 18x - \frac{2x^3}{3}$$

$$v = (18 \times 3) - \frac{2 \times 3^3}{3}$$

$$v = 36$$

To see if its a maximum or minimum value

$$\frac{d^2v}{dx^2} = -4x$$

$$\text{when } x = 3 \quad \frac{d^2v}{dx^2} = -12 \quad \text{as } \frac{d^2v}{dx^2} < 0$$

Volume = 36 is the maximum value