Finding and Using the Common Ratio in a Geometric Sequence

A geometric sequence is a sequence generated by multiplying the previous term by the same number.

The number that you multiply by is called the common ratio.

Example 1. 2, 6, 18, 54 common ratio = 3

$$\frac{6}{2} = 3 \qquad \frac{18}{6} = 3$$

Example 2. $-12,\ 4,\ -\frac{4}{3}$ common ratio = – 1/3

$$\frac{4}{-12} = -\frac{1}{3}$$

To calculate the common ratio find $\frac{u_2}{u_1}$ or $\frac{u_3}{u_2}$

Example 3. If 3, x, 9 are the first 3 terms of a geometric sequence, find the exact value of x

$$\frac{x}{3} = \frac{9}{x}$$

$$(\times 3)\ x = \frac{27}{x}$$

$$(\times x)\ x^2 = 27$$

$$x = \sqrt{27}$$

$$x = 3\sqrt{3}$$

Defining a Geometric Sequence

The general term for a geometric sequence is:-

$u_n = ar^{n-1}$ where a = 1st term,

r = common difference

n = term number

U_n = term you are calculating

$\therefore$ sequence goes a , ar, ar2, ar3, ar4, etc.

Example 1. The first 4 terms of a geometric sequence are 4, 12, 36, 108. Find the 13^{th} term.

$a = 4,\ r = 3,\ n = 13$ Using $u_n = ar^{n-1}$

$$U_{13} = 4 \times 3^{13-1}$$

$$= 4 \times 3^{12}$$

$$= 2125764$$

Example 2. a) Find the common ratio of the geometric sequence 3, 4.5, 6.75, 10.125

$$\text{Common ratio} = \frac{U_{n+1}}{u_n}$$

$$r = \frac{4\cdot5}{3}$$

$$r = 1\cdot5$$

b) Find the 20^{th} term

$a = 3,\ r = 1\cdot5,\ n = 20$ Using $u_n = ar^{n-1}$

$$U_{20} = 3 \times 1\cdot5^{20-1}$$

$$= 3 \times 1\cdot5^{19}$$

$$= 6650\cdot51346$$

c) Find the nth tem

$a = 3,\ r = 1\cdot5,\ n = n$ Using $u_n = ar^{n-1}$

$$U_n = 3 \times 1\cdot5^{n-1}$$

$$= 3 \times \left(\frac{3}{2}\right)^{n-1}$$

Example 3. If the 6th term of a geometric sequence is 32 and the 3rd term is 4. Find the first term and the common ratio.

Using $u_n = ar^{n-1}$

$u_6 = ar^{6-1}$ $\qquad u_3 = ar^{3-1}$

$u_6 = 32$ $\qquad u_3 = 4$

$\therefore \quad ar^5 = 32$ $\qquad ar^2 = 4$

By comparing the ratios

$$\frac{ar^5}{ar^2} = \frac{32}{4}$$

$$r^3 = 8$$

$$\therefore \; r = 2$$

Substitute the value of r into one of your equations

$r = 2 \qquad ar^2 = 4$

$$a \times 2^2 = 4$$

$$4a = 4$$

$$a = 1$$

So the common ratio is 2 and the first term is 1

Growth and Decay Problems

Remember for increases the multiplier is 1.?, if it's a decrease the multiplier is 0.?

Example 1. If property is increasing by 7% per year, what is the multiplier?

$$100\% + 7\% = 107\%$$

$$107\% = \frac{107}{100} = 1 \cdot 07$$

$$\therefore \text{ The multiplier } = 1 \cdot 07$$

Example 2. What is the first term in the geometric progression 3, 6, 12, 24 which will exceed one million?

$a = 3$, $r = 2$ $\quad u_n > 1000000 \quad n = n$

$$\text{using } ar^{n-1} > 1000000$$

$$3 \times 2^{n-1} > 1000000$$

$$2^{n-1} > \frac{1000000}{3}$$

To solve unknown powers we use logs

$$\log 2^{n-1} > \log \frac{1000000}{3}$$

$$(n - 1) \log 2 > \log \frac{1000000}{3}$$

$$n - 1 > \frac{\log 333333 \cdot 33}{\log 2}$$

$$n - 1 > 18 \cdot 35 \ (2dp)$$

$$n > 19 \cdot 35$$

$$\therefore n = 20$$

The 20th term is the first term to exceed 1 million

Finding the Partial Sum of a Geometric Series

General formula for a partial sum is:-

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{where r} > 1$$

or

$$S_n = \frac{a(1 - r^n)}{1 - r} \quad \text{where r} < 1$$

Why?

If $S_n = a + ar + ar^2 + ar^3 + ar^4 + ...ar^{n-1}$

$(\times r)$ $rS_n = ar + ar^2 + ar^3 + ar^4 + ar^5 + ...ar^n$

(subtract equations) $rS_n - S_n = -a + ar^n$ or $S_n - rS_n = a - ar^n$

$S_n(r - 1) = a(r^n - 1)$ $\qquad S_n(1 - r) = a(1 - r^n)$

$S_n = \frac{a(r^n - 1)}{r - 1}$ $\qquad S_n = \frac{a(1 - r^n)}{1 - r}$

Example 1. Find the sum of the first 9 terms of the geometric series 18+12+8+……

$a = 18 \quad r = \frac{12}{18} \quad n = 9$ using $S_n = \frac{a(r^n - 1)}{r - 1}$

$$S_9 = \frac{18(0{\cdot}6666^9 - 1)}{0{\cdot}6666 - 1}$$

$$S_9 = \frac{18(0{\cdot}93988)}{0{\cdot}333333}$$

$$S_9 = 53{\cdot}9945$$

$$S_n = 54\ (2sf)$$

Example 2. Find the sum of the following geometric series 0.5 + 1 + 2 + 4 ……..+1024

$a = 0{\cdot}5 \quad r = 2 \quad u_n = 1024 \quad n = n$ using $u_n = ar^{n-1}$

First we need to know how many terms we are adding together.

$$1024 = 0{\cdot}5 \times 2^{n-1}$$

$$\log 2048 = \log 2^{n-1}$$

$$\log 2048 = (n - 1)\log 2$$

$$\frac{\log 2048}{\log 2} = n - 1$$

$$11 = n - 1$$

$$12 = n$$

This answer means we want to add together the first 12 terms.

$a = 0{\cdot}5 \quad r = 2 \quad n = 12$ using $S_n = \dfrac{a(r^n - 1)}{r - 1}$

$$S_{12} = \frac{0{\cdot}5(2^{12} - 1)}{2 - 1}$$

$$S_{12} = \frac{0{\cdot}5(4095)}{1}$$

$$S_{12} = 2047{\cdot}5$$

$\therefore$ the sum of the first 12 terms is 2047.5

Example 3. Find $\displaystyle\sum_{r=1}^{10} 3 \times 2^r$

Using this information the sequence begins 6, 12, 24 ($3x2^1$, $3x2^2$ etc)

$a = 6 \quad n = 10 \quad r = 2$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{10} = \frac{6(2^{10} - 1)}{2 - 1}$$

$$S_{10} = \frac{6 \times 1023}{1}$$

$$S_{10} = 6138$$

Finding the Sum to Infinity

The total of all the terms in a series is called the sum to infinity. This occurs if r < 1 (which makes it a convergent geometric series and r > -1.

Formula for the sum to infinity:-

$$S_\infty = \frac{a}{1 - r} \quad \text{when} \quad -1 < r < 1$$

Example 1. Find the sum to infinity of the sequence 49, 14, 4

$a = 49 \quad r = \frac{2}{7}$

$\therefore \; S_\infty = \frac{a}{1 - r}$

$= \frac{49}{1 - 0{\cdot}2857}$

$= 68{\cdot}6$

Example 2. Find the first 4 terms of the geometric series if the first term is 12 and the sum to infinity is 24.

$a = 12 \quad S_\infty = 24$ we need to find r

$S_\infty = \frac{a}{1 - r}$

$24 = \frac{12}{1 - r}$

$24(1 - r) = 12$

$24 - 24r = 12$

$12 = 24r$

$0{\cdot}5 = r$

Because r = 0.5

$12 \times 0{\cdot}5 = 6$

$6 \times 0{\cdot}5 = 3$

$3 \times 0{\cdot}5 = 1{\cdot}5$

$\therefore$ first 4 terms are 12, 6, 3, 1.5