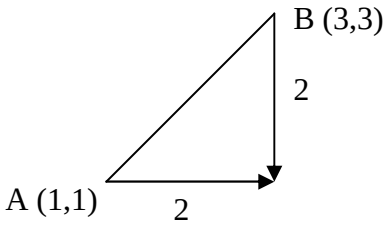
Finding the Midpoint of a Line

To work out the midpoint of line
we need to find the halfway point

Midpoint of AB = (2,2)



The formula for the midpoint is:-

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

Where (x_1, y_1) and (x_2, y_2) are 2 given points on the line

Example 1. If A(3,7) and B(11, − 3) Find the midpoint of AB

$$\text{Midpoint of AB} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

$$= \left(\frac{3 + 11}{2}, \frac{7 + -3}{2} \right)$$

$$= \left(\frac{14}{2}, \frac{4}{2} \right)$$

$$= (7,2)$$

Diameter of circles are often used in this topic because the midpoint will always be the centre of the circle.

Example 2. *If A(2,3) and B is(5,9) and the centre of the circle. If AC is the diameter of the circle find the coordinates of C*

$$\text{Midpoint of AB} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

$$(5, 9) = \left(\frac{2 + x}{2}, \frac{3 + y}{2} \right)$$

$$\therefore\ 5 = \frac{2 + x}{2} \qquad \text{and} \qquad 9 = \frac{3 + y}{2}$$

$$10 = 2 + x \qquad 18 = 3 + y$$

$$8 = x \qquad 15 = y$$

$$\therefore\ C = (5,15)$$

$$\text{if } x = \frac{10}{4} \qquad \text{then} \qquad y = -x - 1$$

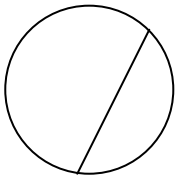
$$y = -\frac{10}{4} - 1$$

$$y = -3{\cdot}5$$

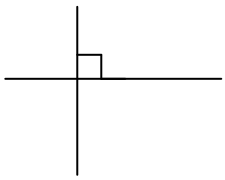
$\therefore$ The Centre of the circle is $(2.5, -3.5)$

Chords and Perpendicular Lines

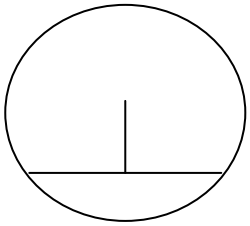
- A chord is a line that passes from one side of a circle to the other but which does not pass through the centre.



- A perpendicular line always cuts at 90°. If it bisects a line then it cuts it exactly in half. It is often called a perpendicular bisector. When questions are talking about this then you need to use the equation of a normal and the midpoints.



- The perpendicular bisector of a chord always passes through the centre of a circle.

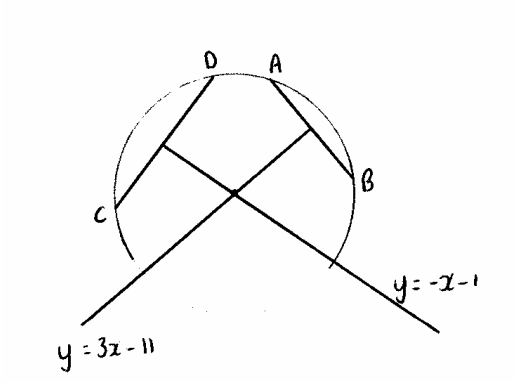


- The key to success is that you always need to draw a sketch so you know what is going on.

Example 1. *The Lines AB and CD are chords of a circle. The line $y = 3x - 11$ is the perpendicular bisector of AB. The line $y = -x - 1$ is the perpendicular bisector of CD. Find the coordinates of the centre of the circle.*

We know the perpendicular bisector of a chord passes through the centre

so the centre of the circle is where the lines meet! So solve simultaneously



$$y = 3x - 11$$

$$y = -x - 1$$

$\therefore \quad 3x - 11 = -x - 1$

$$4x - 11 = -1$$

$$4x = 10$$

$$x = \frac{10}{4}$$

if $x = \frac{10}{4}$ then $y = -x - 1$

$$y = -\frac{10}{4} - 1$$

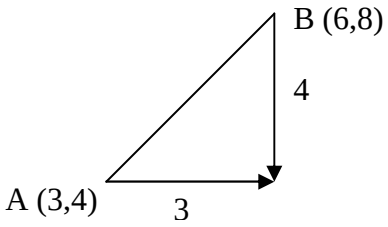
$$y = -3{\cdot}5$$

$\therefore$ The Centre of the circle is $(2.5, -3.5)$

Distance Between Two Points

To work out the distance between two points we use Pythagoras

Midpoint of AB = (2,2)



The formula for the distance between two points is:-

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Where (x_1, y_1) and (x_2, y_2) are 2 given points on the line

Example 1. PQ is the diameter of a circle where p(− 1,3) and Q(6, − 3) .

Find the radius of the circle

First we need to remember that Radius = half the Diameter

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$PQ = \sqrt{(6 - -1)^2 + (-3 - 3)^2}$$

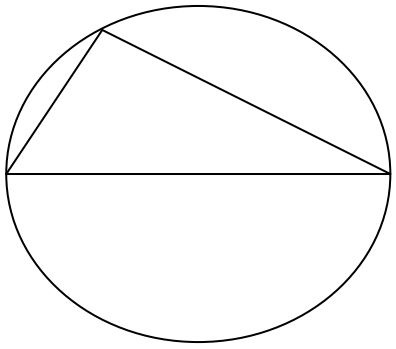
$$PQ = \sqrt{(7^2 + (-6)^2)}$$

$$PQ = \sqrt{85}$$

$$\textbf{Radius} = \frac{\sqrt{85}}{2}$$

Angles in a semicircle

An angle in a semicircle is always 90° when one side of the triangle is the diameter and all 3 sides sit on the circumference of the circle



Example 2. *The points A(2,6) , B(5,7) and C(8, − 2) lie on a circle. Show that ABC is a right angled triangle and find the area of the triangle*

$$\text{Length AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\text{AB} = \sqrt{(5 - 2)^2 + (7 - 6)^2}$$

$$\text{AB} = \sqrt{3^2 + 1^2}$$

$$\text{AB} = \sqrt{10}$$

$$\text{Length BC} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\text{BC} = \sqrt{(8 - 5)^2 + (-2 - 7)^2}$$

$$\text{BC} = \sqrt{3^2 + (-9)^2}$$

$$\text{BC} = \sqrt{90}$$

$$\text{Length AC} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\text{AC} = \sqrt{(8 - 2)^2 + (-2 - 6)^2}$$

$$\text{AC} = \sqrt{6^2 + (-8)^2}$$

$$\text{AC} = \sqrt{100}$$

$$\text{AC} = 10$$

$\therefore$ Using pythagoras to prove ABC is a right angled triangle

$$\text{AC}^2 = \text{AB}^2 + \text{BC}^2$$

$$10^2 = (\sqrt{10})^2 + (\sqrt{90})^2$$

$$100 = 10 + 90$$

$$100 = 100$$

This proves the triangle is a right angled triangle

$$\text{Area of a triangle} = \frac{1}{2} \times \textit{base} \times \textit{height}$$

$$A = \frac{1}{2} \times \text{AB} \times \text{BC}$$

$$A = \frac{1}{2} \times \sqrt{10} \times \sqrt{90}$$

$$A = \frac{1}{2} \times \sqrt{10} \times \sqrt{9} \times \sqrt{10}$$

$$A = 15 \text{ units}^2$$

Equation of a Circle

An equation of a circle is always in the form $(x-a)^2 + (y-b)^2 = r^2$
where r is the radius and (a,b) is the centre of the circle.

Example 1. If a circle has a radius of 7 and a centre at (2,6), what is the equation of the circle?

$(x-a)^2 + (y-b)^2 = r^2$ where a = 2, b = 6, r = 7

$\therefore \quad (x-2)^2 + (y-6)^2 = 7^2$

$(x-2)^2 + (y-6)^2 = 49$

The equation of the cirlce is $(x-2)^2 + (y-6)^2 = 49$

Example 2. Given the equation $(x-2\sqrt{3})^2 + (y+\sqrt{7})^2 = 144$, find the radius of the centre of the circle.

$(x-a)^2 + (y-b)^2 = r^2$ where $a = 2\sqrt{3}$, $b = -\sqrt{7}$, $r^2 = 144$

$\therefore$ centre is $(2\sqrt{3}, -\sqrt{7})$

$r^2 = 144$

$r = \pm 12$ as radius cannot be negative we can ingnore the negative value

$\therefore r = 12$

Example 3. Prove that (1,2) lies on the circumference of the circle which has the equation

$(x-2)^2 + (y+3)^2 = 26$ when x = 1 y = 2

$(1-2)^2 + (2+3)^2 = 26$

$(-1)^2 + (5)^2 = 26$

$1 + 25 = 26$

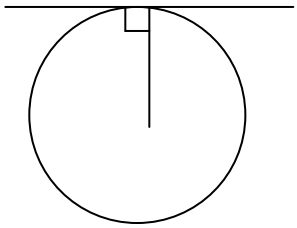
$26 = 26$

$\therefore$ (1,2) lies on the circumference of the circle

Tangents

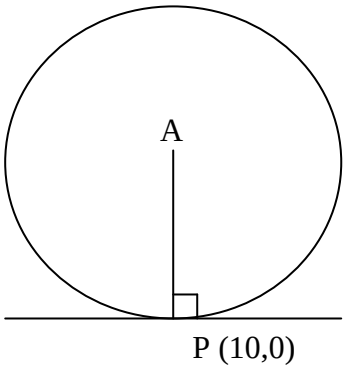
The angle between the tangent and a radius is 90°. A tangent only touches at one point.

This circle theorem is often used in questions as it can relate closely to perpendicular bisectors.



Example 1. The line $4x - 3y - 40 = 0$ touches the circle $(x-2)^2 + (y-6)^2 = 100$ at P(10,0). Show that the radius at P is perpendicular to the line.

This mean the centre A is (2,6)



$$\text{Gradient of AP} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{6-1}{2-10}$$

$$= \frac{6}{-8}$$

$\therefore$ gradient of AP is $-\frac{3}{4}$

Gradient of tangent $4x - 3y - 40 = 0$

$$4x - 40 = 3y$$

$$\frac{4x-40}{3} = y$$

$\therefore$ gradient of tangent is $\frac{4}{3}$

Using $m_1 \times m_2 = -1$ where $m_1 = -\frac{3}{4}$ and $m_2 = \frac{4}{3}$

$$-\frac{3}{4} \times \frac{4}{3} = -1$$

$\therefore$ lines are perpendicualr

Finding Points of Intersection

If you need to find where a circle meets a line then solve the two equations simultaneously.

Example 1. Find where the line $y = x + 5$ meets the circle $x^2 + (y - 2)^2 = 29$

Substitute $y = x + 5$ into $x^2 + (y - 2)^2 = 29$

$$x^2 + ((x + 5) - 2)^2 = 29$$

$$x^2 + (x + 3)^2 = 29$$

$$x^2 + x^2 + 6x + 9 = 29$$

$$2x^2 + 6x - 20 = 0$$

$$x^2 + 3x - 10 = 0$$

$$(x + 5)(x - 2) = 0$$

$$x = -5 \text{ or } x = 2$$

if $x = -5$ $y = x + 5$

$y = -5 + 5$

$y = 0$ $(-5,0)$

if $x = 2$ $y = x + 5$

$y = 2 + 5$

$y = 7$ $(2,7)$

So the line meets the circle at (-5,0) and (2,7).

If you get no solutions when you try and solve two equations then it means the lines do not meet