Definite Integration

So far we have differentiated the gradient function and we have found the function. Now we are going to integrate between 2 values which are called the 'limits of integration'. We are looking for a number as the answer.

Example 1. Evaluate $\int_4^5 2x \, dx$

$= [x^2]_4^5$

$= (5^2) - (4^2)$

$= 25 - 16$

$= 9$

Example 2. Evaluate these definite integrals $\int_1^2 3x^{-2} \, dx$

$= \left[\frac{3x^{-1}}{-1} \right]_1^2$

$= \left[-\frac{3}{x} \right]_1^2$

$= \left(-\frac{3}{2} \right) - \left(-\frac{3}{1} \right)$

$= 1{\cdot}5$

Example 3. Evaluate these definite integrals $\int_0^4 \frac{6}{x^2} \, dx$

$= \int_0^4 6x^{-2} \, dx$

$= \left[\frac{6x^{-1}}{-1} \right]_0^4$

$= \left[-\frac{6}{x} \right]_0^4$

$= \left(-\frac{6}{4} \right) - \left(-\frac{6}{0} \right)$

$= -1{\cdot}5$

Definite integration is defined as

$$\int_b^a f^1(x)\ dx \quad = \quad [f(x)]_a^b \quad = f(b) - f(a)$$

This says you are going to integrate — this shows it is after integration — This is calculating the answer

Example 3. Evaluate $\displaystyle\int_1^3 \frac{x^3 + 2x^2}{x}\ dx$

$$= \int_1^3 \frac{x^3 + 2x^2}{x}\ dx$$

$$= \int_1^3 x^2 + 2x\ \ dx$$

$$= \left[\frac{x^3}{3} + x^2 \right]_1^3$$

$$= \left(\frac{3^2}{3} + 3^2 \right) - \left(\frac{1^3}{3} + 1^2 \right)$$

$$= 18 - 1\frac{1}{3}$$

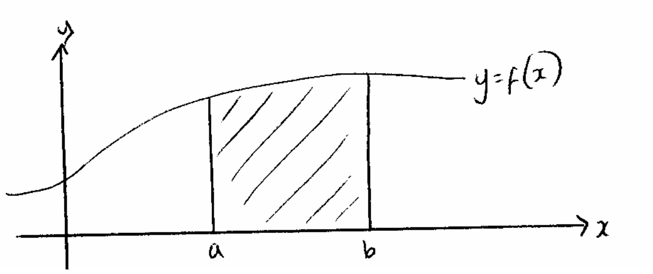
$$= 16\frac{2}{3}$$

Finding Areas Under Curves

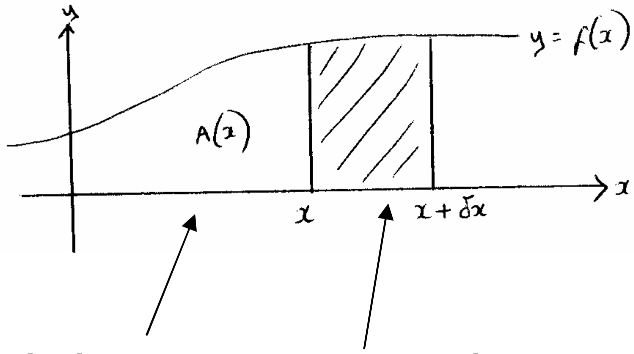
To find the area between a curve, the x-axis and the lines $x=a$ and $x=b$ you have

$$\text{Area} = \int_b^a y \ dx$$

where $y = f(x)$ is the equation of the curve



Why?



This is the area under the curve to the left of x

As x increases, the area increases

If you look at a small increase of x (δx) then the area of the increase would be $A(x + \delta x) - A(\delta x)$.
Because the area is very close to being a rectangle, then the area would be $y \times \delta y$,
as δx becomes smaller, the error of the area of the increase would be smaller also.

So $\delta A \cong y \, dx$

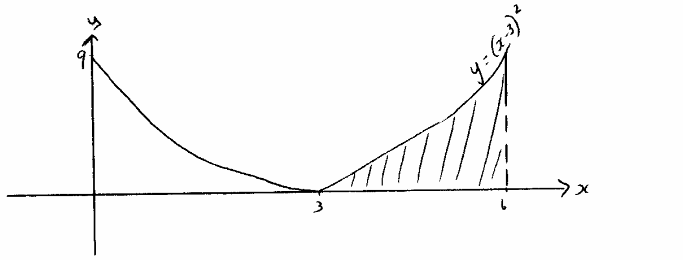
$$\frac{\delta A}{\delta x} \cong y$$ (small changes in Area is approximatley equal to $y \times$ small change in x)

As the limit of δx approaches 0 we can say that $\frac{da}{dx} = y$

$\therefore$ to find A you need to integrate y with respect to x

hence $A = \int y \ dx$

Example 1. Find the area under the curve $y = (x - 3)^2$ between $x = 3$ and $x = 6$.



$$= \int_3^6 (x - 3)^2 \ dx$$

$$= \int_3^6 x^2 - 6x + 9 \ dx$$

$$= \left[\frac{x^3}{3} - 3x^2 + 9x \right]_3^6$$

$$= \left(\frac{216}{3} - 3 \times 6^2 + 9 \times 6 \right) - \left(\frac{27}{3} - 3 \times 3^2 + 9 \times 3 \right)$$

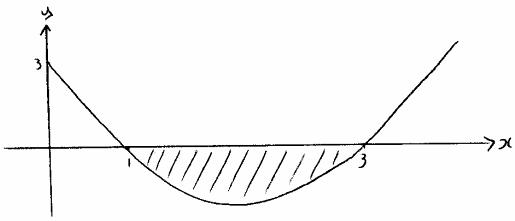
$$= (72 + 108 + 54) - (9 - 27 + 27)$$

$$= 18 - 9$$

$$= 9$$

Calculating Areas of Curves Under the x-axis

Example 1. Find the area of the finite region bounded by the curve $y = x^2 - 4x + 3$ and the x-axis.



$f(x) = x^2 - 4x + 3$

$y = 0 \quad (x - 1)(x - 3) = 0$

$x = 1$ or $x = 3$ — this means it cuts the x axis at 1 and 3

$$\int_1^3 x^2 - 4x + 3 \; dx$$

$$\left[\frac{x^3}{3} - 2x^2 + 3x \right]_1^3$$

$$= \left(\frac{3^3}{3} - 2 \times 3^2 + 3 \times 3 \right) - \left(\frac{1^3}{3} - 2 \times 1^2 + 3 \times 1 \right)$$

$$= (9 - 18 + 9) - \left(\frac{1}{3} - 2 + 3 \right)$$

$$= (0) - \left(1\frac{1}{3} \right)$$

$$= -1\frac{1}{3}$$

$\therefore$ Area $= 1\frac{1}{3}$ — An area cannot be negative so ignore the sign

Example 2. Find the area of the finite region bounded by the curve $y = x^3 - x^2 - 6x$ and the x- axis.

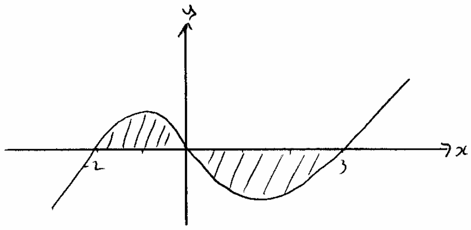
$$y = x^3 - x^2 - 6x$$

$$y = 0 \qquad 0 = x^3 - x^2 - 6x$$

$$0 = x(x^2 - x - 6)$$

$$0 = x(x + 2)(x - 3)$$

$$x = 0 \quad x = -2 \quad x = 3$$



Because there are two areas, deal with them separately

$$\int_{-2}^{0} x^3 - x^2 - 6x \; dx \qquad + \qquad \int_{0}^{3} x^3 - x^2 - 6x \; dx$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{3} - 3x \right]_{-2}^{0} \qquad\qquad = \left[\frac{x^4}{4} - \frac{x^3}{3} - 3x \right]_{0}^{3}$$

$$= (0) - \left(4 - \frac{4}{32} + 6 \right) \qquad\qquad = \left(\frac{81}{4} - 9 - 9 \right) - (0)$$

$$= -8\frac{2}{3} \qquad\qquad = 2\frac{1}{4}$$

$$\therefore \text{ Total area } = 8\frac{2}{3} + 2\frac{1}{4}$$

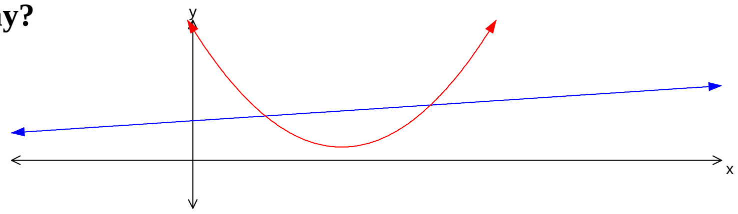
$$= 10\frac{11}{12}$$

Finding the Areas Between a Curves and a Line

The area between a line (equation y_1) and a curve (equation y_2) is given by

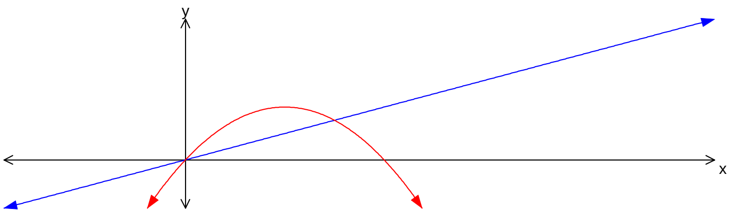
$$\text{Area} = \int_a^b (y_1 - y_2)\ dx$$

Why?



If you consider the area under the straight line (y_1) and take away the answer to the area under the curve (y_2) then you would be left with the area between them.

Example 1. Find the area of the region bounded by the curve $y = x(4 - x)$ and the line $y = x$.



$$x = x(4 - x)$$

$$x = 4x - x^2$$

$$x^2 - 3x = 0$$

$$x(x - 3) = 0$$

$\therefore\ x = 0$ and $x = 3$ this means the points of intersection are 0 and 3

$$= \int_0^3 (y_2 - y_1)\ dx \qquad y_2 - y_1 = x - x(4 - x) = x^2 - 3x$$

$$= \int_0^3 x^2 - 3x\ dx$$

$$= \left[\frac{x^3}{3} - \frac{3x^2}{2} \right]_0^3$$

$$= \left(\frac{3^3}{3} - \frac{3 \times 3^2}{2} \right) - (0)$$

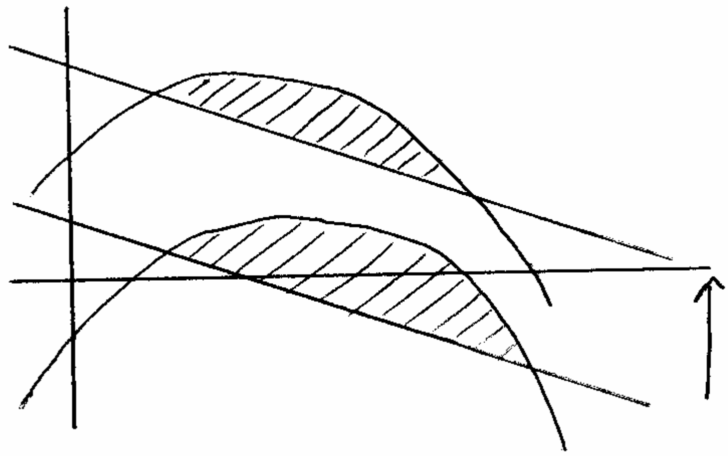
$= -4{\cdot}5 \qquad \therefore$ Area $= 4{\cdot}5$

You may also need to use other area formulas such as:-

$$\text{Triangle} = \frac{1}{2} \times \text{base} \times \text{perpendicular height}$$

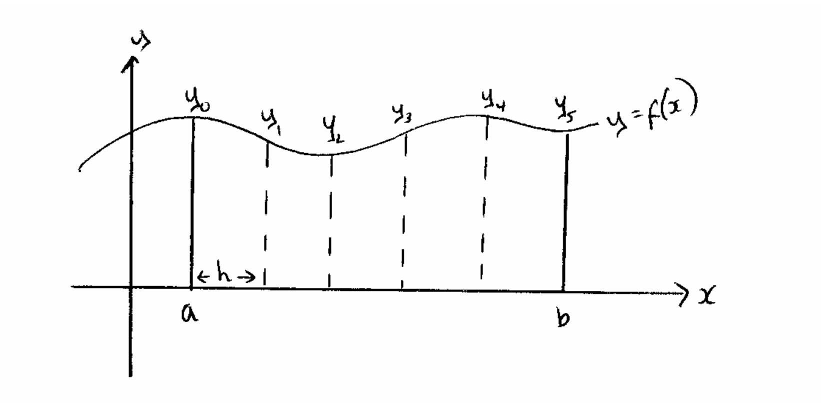
$$\text{Trapezium} = \frac{\text{sum of parallel sides}}{2} \times \text{height}$$

If you have an area that has part of it under the curve then you need to solve it in the same way.



Trapezium Rules

If you want to find the area under a curve but you cannot integrate then you can use the trapezium rule

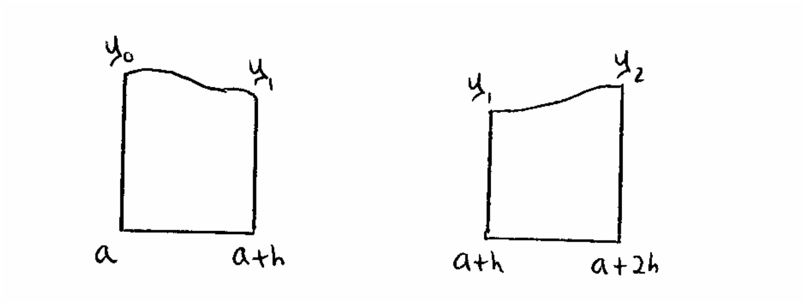


Normally to find the area we would $\int_a^b y \, dx$ instead we divide the area into lots of equal strips and then find the area of each by imagining that they are close to trapeziums.

Step 1. We decide what the width of each trapezium (h) will be by deciding how many strips we are going to have then using this formula

$$h = \frac{b - a}{n} \quad \text{where n is the number of strips}$$

Step 2. now we have the *x* values so we find the corresponding *y* values by substituting it into the original equations, these tell us the heights of the trapeziums.



Step 3. Use the formula to find the area.

Trapezium rule is:- $$\int_a^b y \, dx = \frac{1}{2} h \, [y_0 + 2(y_1 + y_2 + \cdots\cdots y_{n-1}) + y_n]$$

where $h = \dfrac{b - a}{n}$

Always remember the formula for a trapezium $A = \frac{1}{2}(a + b) \times h$

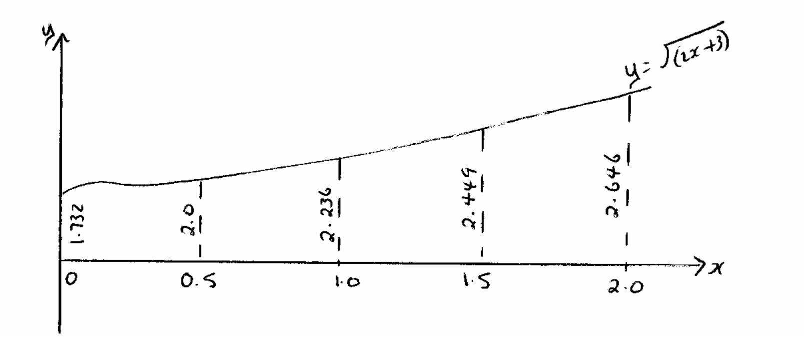
Example 1. Use the trapezium rule with 4 strips to estimate the area under the curve $y = \sqrt{2x + 3}$ between the lines $x = 0$ and $x = 2$

strip width $= \dfrac{b - a}{n}$ where a = 0, b = 2 and n = 4

$$h = \frac{2 - 0}{4}$$

$$h = 0{\cdot}5$$

x	0	0.5	1	1.5	2
y	1.732	2	2.236	2.449	2.646



$$Area = \frac{1}{2} \times 0{\cdot}5 \times [1{\cdot}732 \times 2(2 + 2{\cdot}236 + 2{\cdot}449) + 2{\cdot}646\}$$

$$A = \frac{1}{2} \times 0{\cdot}5 \times 17{\cdot}748$$

$$A = 4{\cdot}437$$

So an estimate for the area is 4.437

(Remember the more strips the more accurate the area)