

Pascal's Triangle

Pascal's Triangle looks like this

$$\begin{array}{cccccc} & & & & & 1 \\ & & & & & & 1 & & 1 \\ & & & & & 1 & & 2 & & 1 \\ & & & & 1 & & 3 & & 3 & & 1 \\ & & 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

When we expand expression such as $(x + y)^4$ we find it follows the same pattern.

Example 1. $(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ Notice the pattern of the coefficients (1,4,6,4,1)

Example 2. $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ Here the coefficients are (1,3,3,1)

The key is to notice the 3 patterns:-

1. The coefficients follow Pascal's triangle sequence
2. The powers of one decrease while the other increase
3. The total power in each expression adds up to the original power in the questions

Example 3. $(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

So now we know this:

$$(a + b)^0 = 1 \quad \text{Line 1}$$

$$(a + b)^1 = 1a + 1b \quad \text{Line 2}$$

$$(a + b)^2 = 1a^2 + 2ab + 1b^2 \quad \text{Line 3}$$

$$(a + b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3 \quad \text{Line 4}$$

$$(a + b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4 \quad \text{Line 5}$$

Example 4. Expand $(2x + 3y)^3$ using Pascal's triangle

$$\begin{aligned} & 4^{\text{th}} \text{ line is } 1,3,3,1 \\ & = 1(2x)^3(3y)^0 + 3(2x)^2(3y)^1 + 3(2x)^1(3y)^2 + 1(2x)^0(3y)^3 \\ & = 8x^3 + (3 \times 4x^2 \times 3y) + (3 \times 2x \times 9y^2) + (1 \times 27y^3) \\ & = 8x^3 + 36x^2y + 54xy^2 + 27y^3 \end{aligned}$$

Example 5.

Fully expand $(1 + 3x)(1 + 2x)^3$

$$\begin{aligned}(1 + 2x)^3 &= 1(1)^3(2x)^0 + 3(1)^2(2x)^1 + 3(1)^1(2x)^2 + 1(1)^0(2x)^3 \\ &= 1 + 6x + 12x^2 + 8x^3\end{aligned}$$

$$\begin{aligned}(1 + 3x)(1 + 2x)^3 &= (1 + 3x)(1 + 6x + 12x^2 + 8x^3) \\ &= 1 + 6x + 12x^2 + 8x^3 + 3x + 18x^2 + 36x^3 + 24x^4 \\ &= 1 + 9x + 30x^2 + 44x^3 + 24x^4\end{aligned}$$

Factorial Notation

A factorial notation looks like this $n! = n \times (n - 1) \times (n - 2) \dots$

Example 1. $8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

To find the number of ways of choosing r items from a group of n items is written as

$${}^n C_r \text{ or } \binom{n}{r}$$

This is calculated by $\frac{n!}{r!(n-r)!}$

Example 2. Find ${}^5 C_3$

$$\begin{aligned} &= \frac{5!}{2! \times 3!} \\ &= \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 3 \times 2 \times 1} \\ &= \frac{120}{2 \times 6} \\ &= 10 \end{aligned}$$

Example 3. Find $\binom{7}{4}$

$$\begin{aligned} &= \frac{7!}{3! \times 4!} \\ &= \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 4 \times 3 \times 2 \times 1} \\ &= 35 \end{aligned}$$

Example 4. 4 People need to sit down but 2 want to sit together. How many different combinations are there?

Using ${}^n C_r = {}^4 C_2$

$$\begin{aligned} &= \frac{4!}{2!2!} \\ &= \frac{4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1} \\ &= 6 \end{aligned}$$

The Connection between Combination Notation and Pascal's Triangle

Look at this:-

$${}^4C_0 = 1 \quad {}^4C_1 = 4 \quad {}^4C_2 = 6 \quad {}^4C_3 = 4 \quad {}^4C_4 = 1$$

These values are the same as the Index 4 line

$$\therefore \text{6th line would be } \begin{pmatrix} 6 \\ 0 \end{pmatrix} \begin{pmatrix} 6 \\ 1 \end{pmatrix} \begin{pmatrix} 6 \\ 2 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \end{pmatrix} \begin{pmatrix} 6 \\ 5 \end{pmatrix} \begin{pmatrix} 6 \\ 6 \end{pmatrix}$$

Using Factorial Notation to work out the Coefficients in the Binomial Expansion

The Binomial Expansion is:-

$$(a + b)^n = (a + b)(a + b)(a + b)\dots\dots(a + b) \\ = {}^n C_0 a^n b^0 + {}^n C_1 a^{n-1} b^1 + {}^n C_2 a^{n-2} b^2 + \dots\dots\dots({}^n C_n) a^{n-n} b^n$$

$$\text{or} \quad \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 \dots\dots\dots \binom{n}{n} a^{n-n} b^n$$

Example 1. Use the binomial theorem to find the expansion of $(3 - 2x)^5$

$$= \binom{5}{0} 3^5 (-2x)^0 + \binom{5}{1} 3^4 (-2x)^1 + \binom{5}{2} 3^3 (-2x)^2 + \binom{5}{3} 3^2 (-2x)^3 \\ + \binom{5}{4} 3^1 (-2x)^4 + \binom{5}{5} 3^0 (-2x)^5 \\ = 243 - 810x + 1080x^2 - 720x^3 + 240x^4 - 32x^5$$

Example 2. a) Write down the first 4 terms of the expansion $\left(1 - \frac{6}{x}\right)^6$

$$= \binom{6}{0} (1)^6 \left(-\frac{x}{10}\right)^0 + \binom{6}{1} (1)^5 \left(-\frac{x}{10}\right)^1 + \binom{6}{2} (1)^4 \left(-\frac{x}{10}\right)^2 + \binom{6}{3} (1)^3 \left(-\frac{x}{10}\right)^3 \\ = 1 + \left(6 \times \frac{x}{10}\right) + \left(15 \times \frac{x^2}{100}\right) + \left(20 \times -\frac{x^3}{1000}\right) \\ = 1 - \frac{6x}{10} + \frac{15x^2}{100} - \frac{20x^3}{1000} \\ = 1 - 0.6x + 0.15x^2 - 0.02x^3$$

- b) By substituting an appropriate value of x , find an approximate value to $(0.99)^6$. Use your calculator to find the degree of accuracy of your approximation.

$$\text{This means we want } \left(1 - \frac{x}{10} \right) = 0.99$$

$$0.01 = \frac{x}{10}$$

$$0.1 = x$$

substitute $x = 0.1$ into $(1 - 0.6x + 0.15x^2 - 0.02x^3)$

$$= 1 - (0.6 \times 0.1) + (0.15 \times 0.1^2) - (0.02 \times 0.1^3)$$

$$= 1 - 0.06 + 0.0015 - 0.00002$$

$$= 0.94148$$

Using a calculator $0.99^6 = 0.941480149$

so approximation is accurate to 5dp as this how far the two answers are the same

Expanding Polynomials Using the Binomial Expansion

You need to remember these facts...

$$(1 + x)^n = \binom{n}{0} 1^n + \binom{n}{1} 1^{n-1} x^1 + \binom{n}{2} 1^{n-2} x^2 \dots\dots\dots\text{etc}$$

$$\text{or} \quad = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} \dots\dots\dots\text{etc}$$

Example 1.


$$\begin{aligned}(1 + 2x)^5 &= 1 + (5 \times 2x) + \frac{5 \times 4 \times (2x)^2}{2!} + \frac{5 \times 4 \times 3 \times (2x)^3}{3!} + \\ &\quad \frac{5 \times 4 \times 3 \times 2 \times (2x)^4}{4!} + \frac{5 \times 4 \times 3 \times 2 \times 1 \times (2x)^5}{5!} \\ &= 1 + 10x + (10 \times 4x^2) + (10 \times 8x^3) + (5 \times 16x^4) + (1 \times 32x^5) \\ &= 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5\end{aligned}$$

Taking Out the Common Factors

The Binomial Expansion only works for $(1 + x)^n$ so sometimes you need to take out a factor first.

$$(2 + x)^3 = \left(2 \left(1 + \frac{x}{2} \right) \right)^3$$

$$= 2^3 \left(1 + \frac{x}{2} \right)^3 \longleftarrow \text{Expand as normal}$$


Remember to
multiply by this at
the end