

Statics of Rigid Bodies

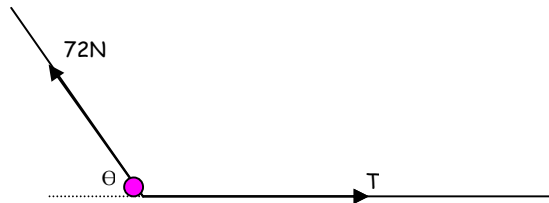
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Recap of M1

It is always best to start with an example from the previous unit to remind you of the basics.

Example 1

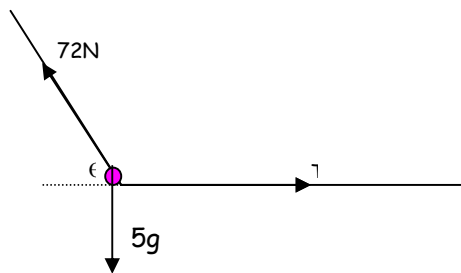
A body of mass 5 kg is held in equilibrium under gravity by two inextensible light ropes. One rope is horizontal and the other is inclined at an angle θ to the horizontal, as shown in the diagram below. The tension in the rope inclined at θ to the horizontal is 72N.



Find

- the angle θ , giving your answer to the nearest degree.
- the tension T in the horizontal rope, giving your answer to the nearest N.

a) Most of the questions from the moments section in M1 can be solved by simply resolving forces. The body is in equilibrium hence the vertical component of the 72N force must equal $5g$.



$$72\sin\theta = 5g \quad (1)$$

$$\theta = 42.88^\circ$$

$$\theta = 43^\circ$$

b) The horizontal component of the 72N force must equal T.

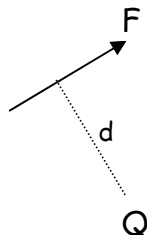
$$72\cos\theta = T$$

$$72 \times \cos 43^\circ = T$$

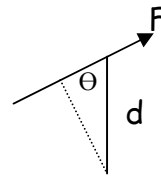
$$T = 53\text{N}$$

The moment of a force about a particular point is:

force \times perpendicular distance.



Moment about Q = $F \times d$



Moment about Q = $Fd\sin\theta$

Remember that the unit is Nm.

Algebraic sum of moments

If a number of coplanar forces act on a rigid body then their moments about a given point may be added. Take great care with the direction of rotation

Equilibrium of Rigid Bodies

A rigid body is said to be in equilibrium if;

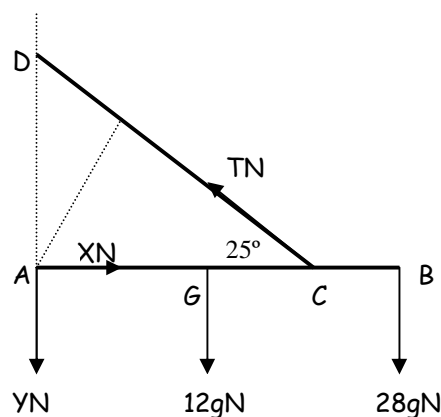
1. the vector sum of the forces acting is zero (the sum of the components in any direction is zero)
2. the algebraic sum of the moments of the force about a particular point is zero.

All problems within this unit need to be attacked in the same fashion.

1. Draw a clearly labeled diagram showing all forces.
2. Resolve the forces perpendicularly and vertically remembering to set them equal to zero. On some occasions the forces will need to be resolved parallel and perpendicular to a plane.
3. Take moments about a convenient point and equate the algebraic sum of the moments to zero. The aim is to choose a point that has only a few unknowns. If you choose a point that makes things more difficult simply select another point.

Example 2

A uniform rod AB of mass 12 kg and length 15 m is smoothly hinged at A and has a particle of mass 28 kg attached to it at B . A light inextensible string is attached to the rod at the point C where $AC = 9\text{m}$ and to the point D vertically above A , keeping the rod in a horizontal position. The tension in the string is $T\text{N}$. If the angle between the rod and the string is 25° , calculate, in terms of T , the resultant moment about A of the forces acting on the rod.



So by taking moments about A, one needs to consider, the weight of the rod, the particle placed at B and the tension in the string acting at C.

$$(12g \times 7.5) + (28g \times 15) - (9 \times T \sin 25^\circ)$$

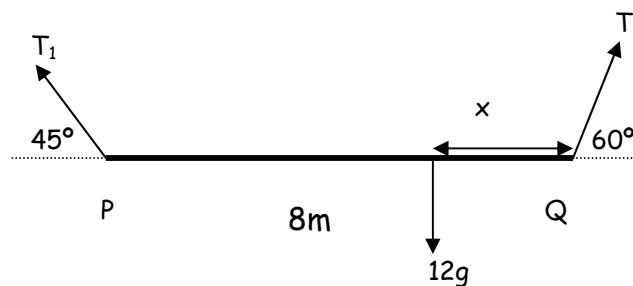
$$(510g - 3.80T)Nm$$

Example 3

A non-uniform rod PQ of mass 12kg and length 8m rests horizontally in equilibrium, supported by two strings attached at the ends P and Q of the rod. The strings make angles of 45° and 60° with the horizontal as shown in the diagram.

(a) Obtain the tensions in each of the strings.

(b) Determine the position of the centre of mass of the rod.



a) Resolving horizontally gives:

$$T_1 \cos 45^\circ = T_2 \cos 60^\circ$$

$$T_1 \frac{\sqrt{2}}{2} = \frac{T_2}{2}$$

$$\sqrt{2}T_1 = T_2$$

Resolving vertically gives:

$$T_1 \sin 45^\circ + T_2 \sin 60^\circ = 12g$$

$$T_1 \frac{\sqrt{2}}{2} + T_2 \frac{\sqrt{3}}{2} = 12g$$

$$\sqrt{2}T_1 + \sqrt{3}T_2 = 24g$$

By substituting back in for T_2 this gives:

$$\sqrt{2}T_1 + \sqrt{2}\sqrt{3}T_1 = 24g$$

$$T_1(\sqrt{2} + \sqrt{6}) = 24g$$

$$T_1 = \frac{24g}{(\sqrt{2} + \sqrt{6})} = 60.87N$$

Therefore $T_2 = 86.09N$

b) Taking moments about Q gives:

$$12gx = 8T_1 \cos 45^\circ$$

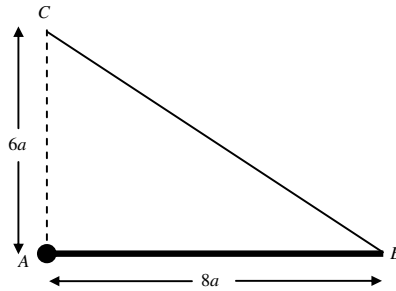
$$12gx = 4T_1 \sqrt{2}$$

$$x = 2.93m$$

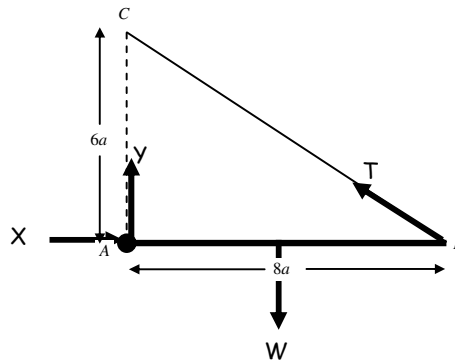
One of the fundamental ideas to remember with this type of problem is that surds must be used until the final calculation. This example may, at first, appear tricky but there will be others to practice on later.

Exam Question

A uniform rod AB , of length $8a$ and weight W , is free to rotate in a vertical plane about a smooth pivot at A . One end of a light inextensible string is attached to B . The other end is attached to point C which is vertically above A , with $AC = 6a$. The rod is in equilibrium with AB horizontal, as shown below.



(a) By taking moments about A , or otherwise, show that the tension in the string is $\frac{5}{6}W$.



Add the forces to the diagram.

Taking moments about A gives:

$$4aW = 8aT \sin B$$

By Pythagoras $CB = 10a$

$$\therefore \sin B = \frac{6a}{10a} = \frac{3}{5}$$

Hence $4aW = 8aT \times \frac{3}{5}$

$$W = 2T \times \frac{3}{5}$$

$$\therefore T = \frac{5}{6}W$$

(b) Calculate the magnitude of the horizontal component of the force exerted by the pivot on the rod.

Resolving forces horizontally.

$$X = T \cos B$$

$$\cos B = \frac{8a}{10a} = \frac{4}{5}$$

$$T = \frac{5}{6}W$$

Therefore

$$X = \frac{2}{3}W$$

Limiting Equilibrium

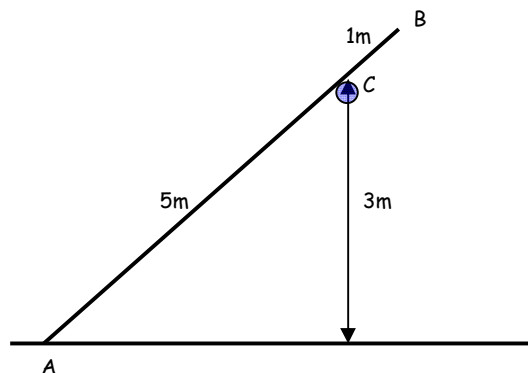
If a body is in limiting equilibrium then one of the forces acting must be friction. The condition to remember is that $F \leq \mu R$. At the instant that motion is about to take place friction will have its highest value of μR .

Example 4

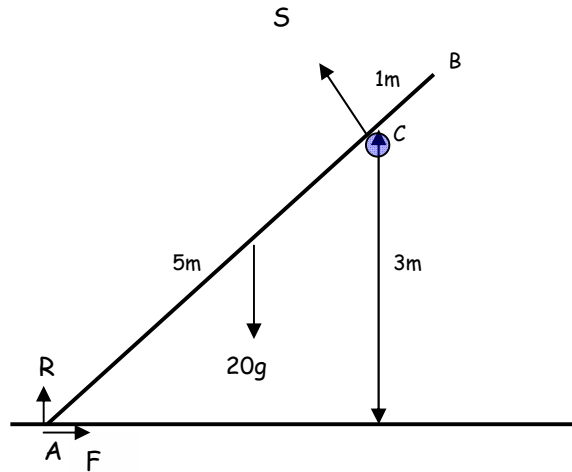
A smooth horizontal rail is fixed at a height of 3m above a horizontal playground whose surface is rough. A straight uniform pole AB, of mass 20kg and length 6m, is placed to rest at point C on the rail with the end A on the playground. The vertical plane containing the pole is at right angles to the rail. The distance AC is 5m and the pole rests in limiting equilibrium.

Calculate:

- the magnitude of the force exerted by the rail on the pole, giving your answer to 3 sig fig.
- the coefficient of friction between the pole and the playground, giving your answer to 2 decimal places.
- the magnitude of the force exerted by the playground on the pole, giving your answer to the nearest N.



a) Adding forces to the diagram:



Taking moments about A gives:

$$5S = 20g \times 3 \times \cos A$$

$$\cos A = 0.8$$

Therefore $S = 94\text{N}$

b) Limiting equilibrium therefore $F = \mu R$.

Resolving horizontally gives:

$$S \times \sin A = F$$

$$S \times 0.6 = F$$

$$F = 56.4\text{N}$$

Resolving vertically gives:

$$S \times \cos A + R = 20g$$

$$94 \times 0.8 + R = 196$$

$$R = 120.8\text{N}$$

$$F = \mu R$$

Therefore $\mu = 0.47$

c) Magnitude of the force exerted by the ground on the pole is given by:

$$= \sqrt{(120.8^2 + 56.4^2)}$$

$$= 133\text{N}$$

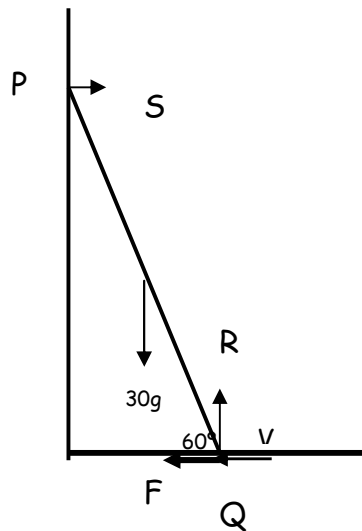
Problems Involving ladders

Ladders will either be lent against a wall or horizontal. We sometimes have to consider frictional forces on the ladder due to the floor or wall (ladder is in contact with a 'rough' surface). Remember that the friction F acts parallel to the surface in such a direction as to oppose the motion.

Example 5

A uniform ladder of mass 30kg rests against a smooth vertical wall with its lower end on rough ground (coefficient of friction 0.25), and its top against a smooth vertical wall. The ladder rests at an angle of 60° to the horizontal. Find the magnitude of the minimum horizontal force required at the base to prevent slipping.

We need to find S , V , R and F .



The easiest one to find first is S .

Taking moments about Q gives:

$$S \times L \sin 60^\circ = 30g \times 0.5L \cos 60^\circ$$

$$S = \frac{30g \times 0.5 \cos 60^\circ}{\sin 60^\circ}$$

$$S = 84.87\text{N}$$

Resolving vertically gives:

$$R = 30g$$

Resolving horizontally gives:

$$S = F + V$$

Given that $F = \mu R$

$$84.87 = 0.25 \times 30 \times g + V$$

$$84.87 = 73.5 + V$$

$$V = 11.4\text{N}$$

What is the maximum horizontal force that could be applied at the base of the ladder without slipping occurring?

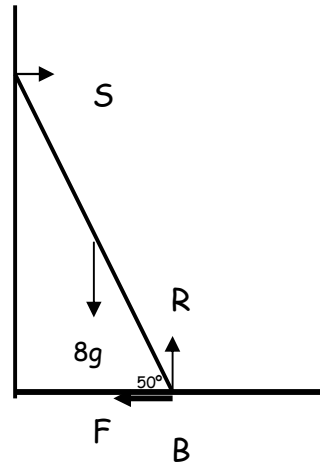
In this situation friction is acting in the opposite direction.

Therefore $S + F = V$

$$84.87 + 73.5 = V \qquad V = 158.4\text{N}$$

Example 6

The diagram shows a ladder AB of mass 8kg and length 6m resting in equilibrium at an angle of 50° to the horizontal with its upper end A against a smooth vertical wall and its lower end B on rough horizontal ground, coefficient of friction μ . Find forces S, F and R and the least possible value of μ if the centre of gravity of the ladder is 2m from B.



Taking moments about B gives:

$$S \times 6 \sin 50^\circ = 8g \times 2 \cos 50^\circ$$

$$S = 21.9\text{N}$$

Resolving horizontally gives:

$$S = F$$

$$F = 21.9\text{N}$$

Resolving vertically gives:

$$R = 8g = 78.4\text{N}$$

Least value of μ occurs when $F = \mu R$

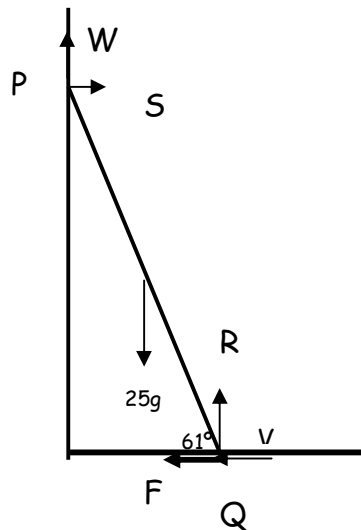
$$\mu = \frac{F}{R}$$

$$\mu = \frac{21.9}{78.4} = 0.28$$

The following example considers friction on the floor and the wall.

Example 7

A uniform ladder of mass 25kg and length L rests against a rough vertical wall (coefficient of friction $\mu = \frac{1}{3}$) with its base on rough ground (coefficient of friction $\mu = \frac{1}{5}$) and it makes an angle of 61° with the ground. Find the magnitude of the minimum horizontal force that must be applied to the base in order to prevent slipping.



Taking moments about Q gives:

$$S \times L \sin 61^\circ + W \times L \cos 61^\circ = 12.5g \times L \times \cos 61^\circ$$

$$S \times \sin 61^\circ + W \times \cos 61^\circ = 12.5g \times \cos 61^\circ \quad (1)$$

Using $W = \mu S$ equation (1) becomes:

$$S \times \sin 61^\circ + \frac{S}{3} \times \cos 61^\circ = 12.5g \times \cos 61^\circ$$

$$S(\sin 61^\circ + \frac{\cos 61^\circ}{3}) = 12.5g \times \cos 61^\circ$$

$$S = \frac{12.5g \times \cos 61^\circ}{\sin 61^\circ + \frac{\cos 61^\circ}{3}} = 57.31\text{N}$$

Resolving vertically gives:

$$W + R = 25g$$

$$\mu S + R = 25g$$

$$R = 25g - \left(\frac{57.31}{3} \right)$$

$$R = 225.9\text{N}$$

Using $F = \mu R$ $F = 0.2 \times 225.9$

$$F = 45.18\text{N}$$

Finally, resolving horizontally gives:

$$F + V = S$$

$$45.18 + V = 57.31$$

$$V = 12.1\text{N}$$

Therefore the minimum horizontal force to prevent slipping is 12.1N

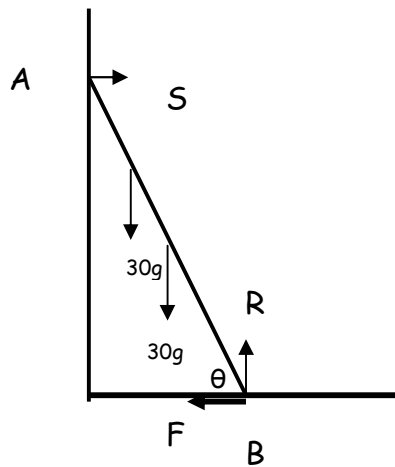
Climbing ladders

Obviously safety is the ultimate concern when climbing a ladder. In deciding whether it is safe to climb to the top of a ladder one has to consider the magnitude of the frictional force acting on the ladder. This in itself is dependent on the roughness of the ground. If a **MAN** is already on a ladder and the system is in limiting equilibrium then any further movement up the ladder will cause it to slip. The example below considers such a situation.

Example 8

A uniform ladder of mass 30kg and length 10m rests against a smooth vertical wall with its lower end on rough ground. The coefficient of friction between the ground and the ladder is 0.3. The ladder is inclined at an angle θ to the horizontal where $\tan \theta = 2$. Find how far a boy of mass 30 kg can ascend the ladder without it slipping.

Assume that the boy can climb a height y m up the ladder.



Taking moments about B gives:

$$S \times 10 \times \sin \theta = 30g \times y \times \cos \theta + 30g \times 5 \times \cos \theta$$

$$10S \times \sin \theta = 30g(y + 5)\cos \theta$$

$$10S \times \tan \theta = 30g(y + 5)$$

$$20S = 30g(y + 5)$$

Resolving vertically gives:

$$R = 60g$$

Given that $F = \mu R$, $F = 0.3 \times 60g = 18g$

Resolving horizontally gives:

$$F = S$$

Therefore $S = 18g$

Using $20S = 30g(y + 5)$

$$20 \times 18g = 30g(y + 5)$$

$$12 = y + 5$$

$$y = 7$$

Therefore the boy can climb 7m up the ladder.

Questions

1 A uniform rod of mass M rests in limiting equilibrium with the end A standing on rough horizontal ground and the end B resting against a smooth vertical wall. The vertical plane containing AB is perpendicular to the wall. The coefficient of friction between the rod and the ground is 0.2 . Find, to the nearest degree, the angle at which the rod is inclined to the vertical.

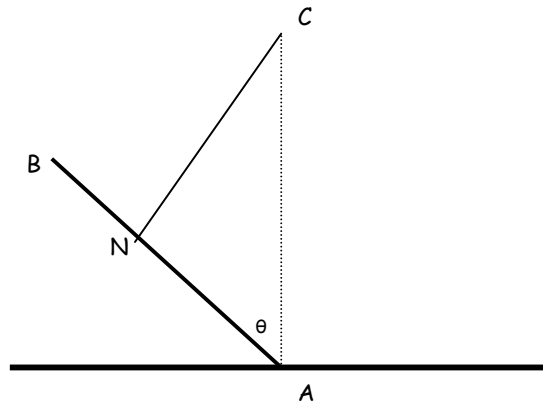
2 A uniform rod of mass M rests in limiting equilibrium with the end A standing on rough horizontal ground and the end B resting against a smooth vertical wall. The vertical plane containing AB is perpendicular to the wall. The coefficient of friction between the rod and the ground is 0.75 . Given that the ladder makes an angle α with the floor show that

$$\tan\alpha = \frac{2}{3}$$

3 A non uniform ladder AB of length 15m and mass 40kg has its centre of gravity at a point 5m from A . The ladder rests with end A on rough horizontal ground (coefficient of friction 0.25) and end B against a rough vertical wall (coefficient of friction 0.2). The ladder makes an angle α with

the horizontal such that $\tan\alpha = \frac{9}{4}$. A straight horizontal string connects A to a point at the base of the wall directly below B . A man of mass 80kg begins to climb the ladder. How far up the ladder can the man climb without causing tension in the string? What tension must the string be capable of withstanding if the man is to climb to the top of the ladder?

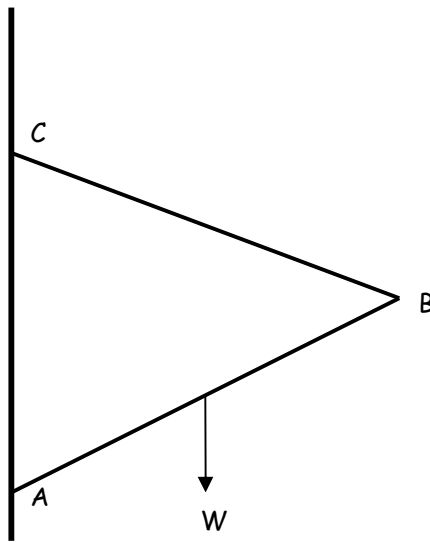
4 The diagram below shows a uniform ladder AB of length $2a$ and mass m , with the end A resting on a rough horizontal floor.



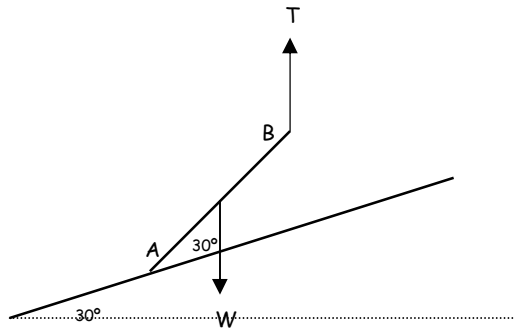
The ladder is held at an angle θ to the vertical by means of a light inextensible rope attached to the point N, where $AN = 1.5a$. The other end of the rope is attached to a point c, which is at a height $3a$ vertically above the end A of the ladder. By taking moments about C find the magnitude of the force of friction acting on the ladder at A. Also calculate the magnitude of the vertical component of the reaction at A.

Given that the coefficient between the floor and the ladder is $\frac{1}{\sqrt{3}}$, show that when the ladder is on the point of slipping at A its inclination to the vertical is given by $\theta = \frac{\pi}{3}$.

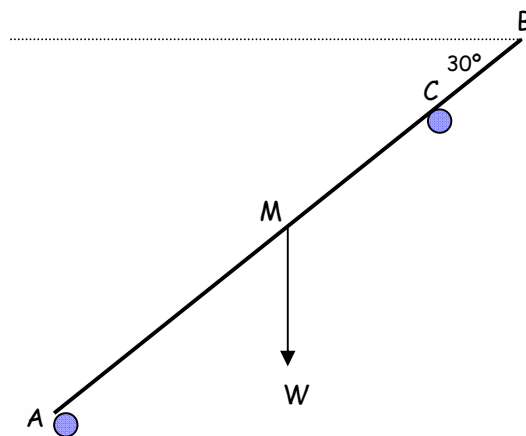
- 5 The figure shows a uniform rod AB of weight W resting with one end A against a rough vertical wall. One end of a light inextensible string is attached at B and the other end is attached at a point C , vertically above A . The points A , B and C lie in the same vertical plane with $AB = BC = 4a$ and $AC = a$. If the system is in limiting equilibrium, calculate:
- the tension in the string
 - the angle that the rod makes with the horizontal.
 - the magnitude of the resultant force acting at A .



6 The diagram below shows a uniform rod AB , of weight W being held in limiting equilibrium at an angle of 30° to a rough plane by a vertical string at the point B . The end A is in contact with a rough surface inclined at an angle of 30° to the horizontal. Find T in terms of W and calculate the coefficient of friction between the rod and the inclined plane.

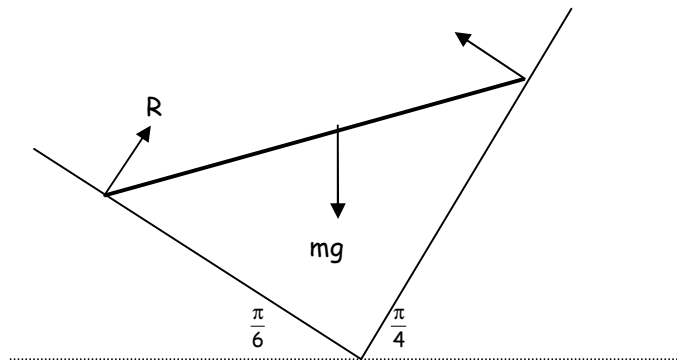


7 The diagram below shows a uniform rod at rest in limiting equilibrium on a rough peg at A and a smooth peg at C . Given that $MC = CB$, find the coefficient of friction at A and the normal reaction at C (in terms of W).



Extension

1 The diagram below shows a heavy uniform rod of length $2a$ and mass m , resting in equilibrium with its two ends on two smooth surfaces. The normal reactions at the ends of the rods have magnitudes R and S . The inclinations of the planes to the horizontal are $\frac{\pi}{6}$ and $\frac{\pi}{4}$ and the rod lies in the vertical plane containing lines of greatest slope of both planes.



a) Show that $R = S\sqrt{2}$.

b) By taking moments about the centre of the rod, prove that the inclination of the rod to the horizontal is $\text{Cot}^{-1}(1 + \sqrt{3})$. The identities for $\text{Cos}(A + B)$ and $\text{Cos}(A - B)$ may be useful.