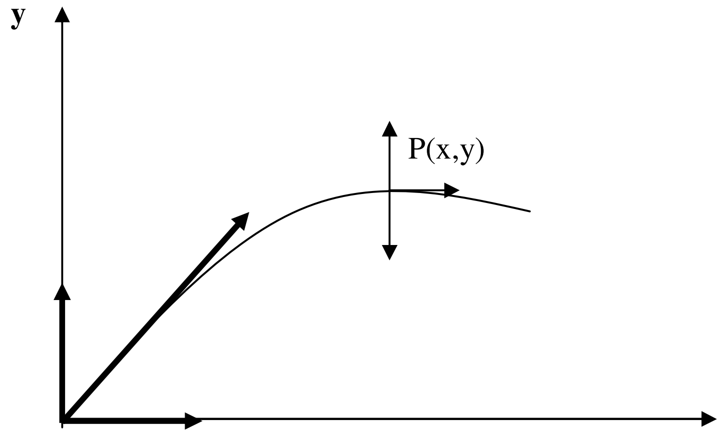
Kinematics of a Particle Moving in a Straight Line or Plane

Projectiles

When a body is projected from a point in such a way that the only force assumed to be acting is gravity then that body is classed as a projectile.

Parametric and Cartesian forms of equations of the trajectory (flight path)

The derivation of the following equations is a little tricky but the process has appeared on an Edexcel paper at M2.



Suppose a particle P is projected from O at angle α and is at the point (x,y) at time t after leaving O.

Equation of motion for P horizontally is F = ma

$$0 = m\ddot{x}$$

Therefore $\ddot{x} = 0$ and velocity parallel to Ox is constant and equal to $V\cos\alpha$.

For constant velocity $x = \text{vel} \times \text{time}$

$$x = V\cos\alpha t \qquad (1)$$

Equation of motion for P vertically F = ma

$$-mg = m\ddot{y}$$
$$\ddot{y} = -g$$

Using constant acceleration equations $s = ut + \frac{1}{2}at^2$

With u = sinα and a = -g $y = V\sin\alpha t - \frac{1}{2}gt^2$ (2)

1 and 2 are the parametric equations of trajectory.

Rearrange 1 to give:

$$t = \frac{x}{V\cos\alpha}$$

Substitute into equation 2

$$y = V\sin\alpha \times \frac{x}{V\cos\alpha} - \frac{1}{2}g\frac{x^2}{V^2\cos^2\alpha}$$

$$y = x\tan\alpha - \frac{g}{2V^2}x^2\sec^2\alpha \qquad (3)$$

Equation 3 is the Cartesian form of the trajectory of a projectile. Assuming V, g and α are constants for any particular case, this is the equation of a parabola.

Maximum height of a projectile.

At the maximum height the vertical component of the velocity is zero.

$$V^2 = u^2 + 2as$$

$$0 = (V \sin\alpha)^2 - 2gH$$

$$H = \frac{V^2 \sin^2 \alpha}{2g}$$

Time to maximum height and time of flight

Using $v = u + at$

At maximum height $v = 0$

$u = V\sin\alpha$ $\quad 0 = V\sin\alpha - gt$

$$t = \frac{V\sin\alpha}{g}$$

For time of flight on the horizontal plane we use:

$$s = ut + \frac{1}{2}at^2$$

There is zero displacement vertically hence s = 0.

$$0 = V\sin\alpha t - \frac{1}{2}gt^2$$

$$0 = t\left(V\sin\alpha - \frac{1}{2}gt\right)$$

$$\therefore t = 0$$

and

$$t = \frac{2V\sin\alpha}{g}$$

By symmetry this time is twice the value to maximum height.
For sloping planes, (cliffs etc) use s as the vertical displacement from the starting point to the landing point.

Range on the horizontal plane

Using equation 1 $x = V\cos\alpha t$

Time in flight is $t = \dfrac{2V\sin\alpha}{g}$

Therefore range R is: - $R = V\cos\alpha \times \dfrac{2V\sin\alpha}{g}$

$$R = \frac{2V^2 \sin\alpha\cos\alpha}{g}$$

$2\sin\alpha\cos\alpha = \sin 2\alpha$

$$R = \frac{V^2 \sin 2\alpha}{g}$$

Maximum range appears when $\sin 2\alpha = 1$
Therefore $\alpha = 45°$ and

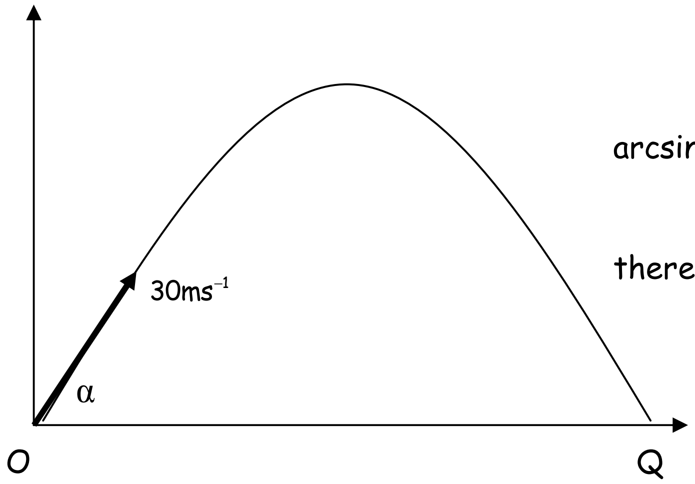
$$R_{Max} = \frac{V^2}{g}$$

Examples

1. A particle is projected from a point O with a speed of 30ms^{-1} at an angle of elevation of $\arcsin\left(\frac{3}{5}\right)$.

a) Find the greatest height above O reached by the particle .

b) The particle strikes the horizontal through O at Q find the distance OQ.



arcsin$\left(\frac{3}{5}\right)$ means sin α= $\frac{3}{5}$

therefore cos α = $\left(\frac{4}{5}\right)$

a) Using the maximum height formula

$$H = \frac{V^2 \sin^2 \alpha}{2g}$$

$$H = \frac{30^2 \times \left(\frac{3}{5}\right)^2}{2g}$$

$$H = 16.5m$$

You could also work out the answer by considering the vertical component of the velocity and then use the constant acceleration equations.
Vertical component is given by

$$V \sin \alpha = 30 \times \frac{3}{5}$$

At maximum height velocity equals zero.
Using:

$$V^2 = u^2 + 2as$$

$$0 = 18^2 - 2gs$$

$$s = \frac{18^2}{2g}$$

$$s = 16.5m$$

b) OQ is the horizontal range

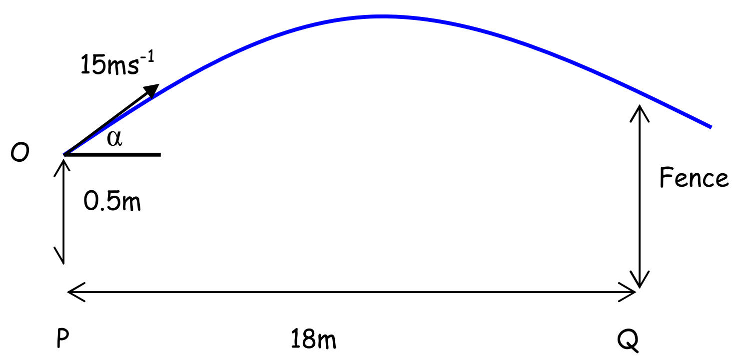
$$R = \frac{V^2 \sin 2\alpha}{g}$$

Which can be expressed as:

$$R = \frac{2V^2 \sin\alpha \cos\alpha}{g}$$

$$R = \frac{2 \times 30^2 \times \frac{3}{5} \times \frac{4}{5}}{9.8}$$ $$R = 88.2m$$

2. A girl hits a ball at an angle $\arctan\left(\frac{3}{4}\right)$ to the horizontal from a point O which is 0.5m above level ground. The initial speed of the ball is $15ms^{-1}$. The ball just clears a fence which is a horizontal distance of 18m from the girl. By modelling the ball as a particle find the time taken for the ball to reach the fence and the height of the fence.



Horizontal component of the velocity is Vcosα.
Therefore time in flight is given by:

$$t = \frac{d}{v}$$

$$t = \frac{18}{V\cos\alpha} = \frac{18}{15 \times \frac{4}{5}}$$

$$t = 1.5 sec$$

We must now consider the motion vertically to calculate the height of the fence. The vertical component of the velocity is:

$V \sin\alpha = 15 \times \frac{3}{5} = 9ms^{-1}$ and by using $s = ut + \frac{1}{2}at^2$

$$s = 9 \times 1.5 - \frac{1}{2} \times (1.5)^2$$
$$s = 12.375m$$

Fence height = 12.9m

Questions A

1 A cricket ball is hit from a point which is 0.9m above horizontal ground. It is given an initial speed of 13ms^{-1} at an angle of elevation of 27 °. Find:

a) the time taken for the ball to reach the ground.

b) the horizontal distance covered by the ball.

2 A tennis ball is served from a height of 2.6m at horizontal speed of 22ms^{-1}. The net is 0.9m high and 13m horizontally from the server. Modelling the ball as a particle, determine whether the ball clears the net and if so by what distance.

3 A golfer hits a ball with velocity 50ms^{-1} at an angle θ above the horizontal, where $\tan\theta = \left(\frac{5}{12}\right)$. Find the time for which the ball is at least 12m above the ground.

4 A particle is projected from a point O with speed 60ms^{-1} at an angle $\cos^{-1}\frac{4}{5}$ above the horizontal. Find

a) the time the particle takes to reach the point P whose horizontal displacement is 96 metres,

b) the height of P above O,

c) the speed of the particle 2 seconds after projection.

5 A cannon ball fired at an angle of 10° has a range, on a horizontal plane, of 1.25km. Ignoring air resistance, find the speed of projection.

6 A ball is projected with velocity 25ms^{-1}. If the range on the horizontal plane is 60m, find the two possible angles of projection.

7 At time t seconds, where t≥0, the velocity $v\,\text{ms}^{-1}$ of a particle Q moving in a straight line is given by

$$v = 8t - 4t^2$$

When t = 0, Q is at point O.

a) Find the acceleration of Q at time t.

b) Calculate the time at which Q returns to O.

Differentiating and Integrating Vectors

The constant acceleration equations were used extensively in M1 but in M2 displacement is a function of time. It follows then that velocity, which is the rate of change of displacement, can be found by differentiating the displacement function.

Differentiate ↓	Displacement	↑ Integrate
	Velocity	(Remember the constant)
	Acceleration	

If **r** is the position vector, and using dot notation: -

$$\dot{r} = \frac{dr}{dt} = velocity$$

$$\ddot{r} = \frac{d^2r}{dt^2} = acceleration$$

Example 1

The position vector of a particle Q at time *t* is $r = (3t+1)i + 2t^2 j$
(r measured in metres).
Find the initial position vector and show that the acceleration is constant

Initial position is when *t=0*

$$r = i$$

Remembering that $\ddot{r} = acceleration$

$$\dot{r} = 3i + 4tj$$

$$\ddot{r} = 4j$$

Since the acceleration vector has no variable *t* it is said to be constant.

Integrating Vectors

Since a vector has both *i* and *j* components it will also have two separate functions with respect to time.

Therefore if $\ddot{r} = f(t)i + g(t)j$

We need to integrate each function separately, remembering the constants.

Therefore $\dot{r} = i\int f(t)dt + j\int g(t)dt + C_1 i + C_2 j$

Where C_1, C_2 are constants of integration.
R is found by integrating the velocity function with respect to time.

Example 2
A particle moves such that at time ***t***

$$\dot{r} = 4ti + 5t^2$$

At time t = 0 the particle has a position vector 5***i*** – 6***j***
Find the position vector at time *t.*

The position vector is found by integrating the velocity vector.

$$r = i\int 4t\,dt + j\int 5t^2 dt + C_1 i + C_2 j$$

$$r = 2t^2 i + \frac{5}{3}t^3 j + C_1 i + C_2 j$$

At time t = 0, r = 5***i*** – 6***j*** $\quad \therefore \quad C_1 = 5, C_2 = -6$

$$\therefore \quad r = (2t^2 + 5)i + \left(\frac{5}{3}t^3 - 6\right)j$$

Example 3
A particle Q has position vector (25***i*** – 40***j***)m at time t = 0 relative to the origin. Q moves with constant acceleration and is equal to (5i + 12j)ms^{-2} .

When t = 0, $\dot{r} = 0$. Find:

a) $\dot{r}$ when t = 4
b) The distance of Q from O at this time.

a) By integrating the acceleration vector we can find the velocity vector.

$$\ddot{r} = 5i + 12j$$

$$\therefore \dot{r} = i\int 5dt + j\int 12dt + C_1 i + C_2 j$$

$$\dot{r} = 5ti + 12tj + C_1 i + C_2 j$$

At time t = 0, $\dot{r} = 0$

$$C_1 i = C_2 j = 0$$

$$\therefore \dot{r} = 5ti + 12tj$$

When t = 4

$$\dot{r} = (20i + 48j)ms^{-1}$$

b) To find the distance OQ we need the position vector (r)

$$\dot{r} = 5ti + 12tj$$

$$\therefore r = i\int 5tdt + j\int 12tdt + K_1 i + K_2 j$$

$$r = \frac{5}{2}t^2 i + 6t^2 j + K_1 i + K_2 j$$

When t = 0, r = 25i – 40j

$$\therefore r = (\frac{5}{2}t^2 + 25)i + (6t^2 - 40)j$$

Substituting t = 4 gives

$$r = 65i + 56j$$

Distance OQ is the magnitude of r

$$OQ = |r|$$
$$= 85.8m$$

Example 4

A remote control car is being tested in a horizontal playground. At time t seconds, the position vector, **r** metres, of the car relative to a fixed point O is given by

$$r = \frac{9}{2}t^2 i + \frac{8}{5}t^{\frac{5}{2}} j$$

At the instant when t = 4,

a) show that the car is moving with velocity $(36i + 32j)ms^{-1}$

b) find the magnitude of the acceleration of the car.

A cyclist is moving with constant velocity $17jms^{-1}$. At the instant when t = 4, calculate

c) the velocity of the car relative to the cyclist;

d) the speed of the car relative to the cyclist;

e) the acute angle between the relative velocity and the constant velocity of the cyclist.

a) $Velocity = \frac{dr}{dt}$

$$r = \frac{9}{2}t^2 i + \frac{8}{5}t^{\frac{5}{2}} j$$

$$v = 9ti + 4t^{\frac{3}{2}} j$$

When t = 4 $\quad Velocity = (36i + 32j)ms^{-1}$

b) $Acceleration = \frac{dv}{dt}$

$$\frac{dv}{dt} = 9i + 6t^{\frac{1}{2}} j$$

When t = 4 $\quad a = 9i + 12j$

$$|a| = \sqrt{(9^2 + 12^2)}$$

So $\quad |a| = 15ms^{-2}$

c) The velocity of the car relative to the cyclist is given by:

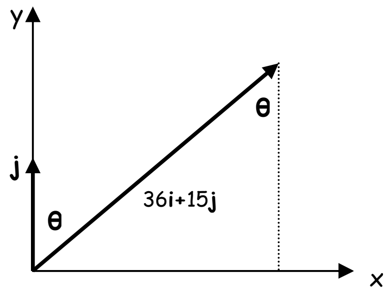
$$v_{car} - v_{cyclist} = 36i + 32j - 17j = 36i + 15j$$

d) the speed of the car relative to the cyclist is given by:

$$\left|v_{car} - v_{cyclist}\right| = \sqrt{(36^2 + 15^2)}$$

$$= 39ms^{-1}$$

e) The cyclist is moving parallel to the vector **j**.
The car has velocity $(36i + 32j)ms^{-1}$ which can be represented diagrammatically as:



The required angle is θ, where

$$\tan\theta = \frac{36}{15}$$

$$\theta = 67.4^{\circ}$$

Example 5
A particle P of mass 5kg is acted on by a constant force **F**. At time t seconds the position of the particle, **r** metres, is given by the equation

$$r = (3t^2 - kt + 2)i + (kt^2 + 4t - k)j$$

where k is a positive constant.

a) Find the acceleration of P in ms^{-2} in terms of k
b) Given that the magnitude of **F** is 50N, calculate the value of k.

The particle is moving in a direction parallel to **j** when t = T.

c) Find the value of T.
d) Hence find, to the nearest 0.1°, the angle that the position vector makes with the direction of **i** when t = T

a) To find the acceleration we need to differentiate the position vector twice.

$$r = (3t^2 - kt + 2)i + (kt^2 + 4t - k)j$$

$$\frac{dr}{dt} = (6t - k)i + (2kt + 4)j$$

$$\frac{d^2r}{dt^2} = 6i + 2kj$$

b) F = 50N and by using F = ma

$$F = 5(6i + 2kj)$$

$$|F| = \sqrt{(30^2 + 10^2 k^2)}$$

$$|F| = 10\sqrt{(9 + k^2)}$$

$$|F| = 50$$

$$\therefore 5 = \sqrt{(9 + k^2)}$$

$$K^2 = 16$$

$$K = \pm 4$$

But K is positive, so K = 4

c) The particle is now traveling parallel to **j**. Therefore the **i** component of the velocity must equate to zero.

$$v = (6t - k)i + (2kt + 4)j$$

Therefore:

$$6t - 4 = 0$$

$$t = \frac{2}{3}$$

d) When $t = \frac{2}{3}$

$$r = (3t^2 - 4t + 2)i + (4t^2 + 4t - k)j$$

$$r = \frac{2}{3}i + \frac{4}{9}j$$

The angle that the position vector makes with the vector **i** is given by:

$$\tan\theta = \frac{\frac{4}{9}}{\frac{2}{3}}$$

$$\theta = 33.7^\circ$$

Questions B

1 At time t=0 a particle Q is at the point with position vector (6**i** + 10**j**)m relative to a fixed origin O. The particle moves with constant acceleration $\ddot{r}$ where $\ddot{r} = (5i + 12j)ms^{-2}$. Given that when t=0, $\dot{r} = 0$ find

a) the velocity, $\dot{r}$, when t=3,
b) the distance of Q from O at this time.

2 A particle Q moves such that at time t seconds, t≥0, its position vector, **r** metres, relative to a fixed origin is given by

$$r = (3t - t^3 + 2)i + (t^2 + 2t)j$$

a) Find the velocity of Q when t = 3.
The velocity of Q is parallel to (3**i** – **j**) when t = T.
b) Find the value of T.

3 A particle moves so that at time t seconds its position vector, **r**m, relative to a fixed origin is given by:

$$r = (t^2 - 4t)i + (t^3 + bt^2)j$$

where b is a constant.
a) Find an expression for the velocity of the particle at time t seconds.
b) Given that the particle comes to instantaneous rest, find the value of b.

4 A ball of mass 0.1kg is hit by a bat which gives it an impulse of (3.5**i** + 3**j**)Ns. The velocity of the ball immediately after being hit is $(10i + 25j)ms^{-1}$.

a) Find the velocity of the ball immediately before it was hit.

In the subsequent motion the ball is modeled as a particle moving freely under gravity. When it is hit the ball is 0.8m above the ground.

b) Find the greatest height of the ball above the ground.

The ball is caught when it is 0.8m above the ground again.

c) Find the distance from the point where the ball is hit to the point where it is caught.